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Analytical approximations of the shape factors for conductive heat flow in circular and regular polygonal cross-sections

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Abstract

The heat flux in steady heat conduction through cylinders whose cross-section has an inner or an outer contour in the form of a regular polygon or a circle is considered. To calculate the shape factor the temperature field is determined. Three cases are considered: (a) temperature field for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons, (b) temperature field for hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles, (c) temperature field for hollow prismatic cylinders bounded by isothermal inner and outer regular polygons. The boundary collocation method in the least squares sense for solving appropriate boundary value problems is used. By means of nonlinear approximation (Marquardt method), for the three considered geometry cases, the simple analytical formulas for the shape factors are proposed.

Keywords: Geometry; Heat trasfer; Heat exchangers; Numerical methods

1. Introduction

Solid shells of a regular shaped cross-section, where inner and outer boundaries have different but constant temperatures find use in many industry processes. For heat transfer purposes it is of interest to know the heat flow rate through the walls of shells. A very efficient way to calculate the heat transfer in these cases is the use of the shape factor, probably first introduced by Langmuir [1]. Consider the cross-section where the two boundaries Γ_i and Γ_o possess two different but constant temperatures T_i and T_o (see Figs. 1, 3 and 5). The steady-state heat transfer rate per unit length, Q_l can be expressed as:

$$Q_l = \lambda S(T_o - T_i), \tag{1}$$

where S is Langmuir's shape factor, defined as

$$S = \int_{\Gamma} \frac{\partial \Theta}{\partial \mathbf{n}} \, d\Gamma, \tag{2}$$

whereas Θ is dimensionless temperature, defined as

$$\Theta = \frac{T - T_o}{T_i - T_o}. \tag{3}$$

Once S is known for a two-dimensional shape with constant heat conductivity and each boundary at uniform temperature, the total heat flow may be easily calculated by the use of Eq. (1).

The subject of this paper are the simple analytical formulas for the shape factors for cylinders whose cross-section has an inner or an outer contour in the form of a regular polygon or a circle. Such subject in literature is not new. An extensive review of papers published before 1972 related with formulas not only for shape factor, but also for geometry with regular polygons, one can find this in [2]. Taking into account the shape factor for domains with regular polygonal shapes, first of all in the book on heat transfer (for e.g. [3], p. 75) one can find shape factor for cross-section bounded by two circles in the form

$$S = \frac{2\pi}{\ln\left(\frac{1}{E_{cc}}\right)}. \tag{4}$$

This formula can be considered as the shape factor for limited case when number of polygonal sides L (inner or outer boundary) tends to infinity.

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Nomenclature

a outer radius (m)
b inner radius (m)
$A, B, C, D, A_k, B_k, C_k, D_k, \lambda_k$
integral constants in general solution (14)
BCMLSS
abbreviation of the boundary collocation method in the least squares sense
E_{cc} ratio of the inner radius of a tube to its outer radius
E_{cp} ratio of the inner radius of circular boudary b to the radius of the inscribed circle in regular polygonal outer boundary a
E_{pc} ratio of the radius of the circumscribed circle on inner regular polygonal boundary b to the radius of outer circular boundary a
E_{pp} ratio of the radius of the circumscribed circle on inner regular polygonal boundary b to the radius of circumscribed circle on outer regular polygonal boundary a
F_j right-hand vector in a system of linear equations resulting from simple collocation
G_{jk} matrix in a system of linear equation resulting from simple collocation
H_{jk} matrix in a system of linear equations resulting from collocation in the least squares sense
$k_1, k_2,$ and k_3
coefficients in formula (83)
L number of polygonal sides
$\mathbf{n}$ unit vector perpendicular to a surface (m)
P_j right-hand vector in a system of linear equations resulting from collocation in the least squares sense
Q_l heat flow per unit length (W/m)
r polar coordinate (m)
R nondimensional polar coordinate
S shape factor (Eq. (1))
SBCM abbreviation of the simple boundary collocation method
T temperature (K)
x Cartesian coordinate (m)
X nondimensional Cartesian coordinate
Y_k unknowns in linear systems

Greek symbols
Γ contour of boundary
λ thermal conductivity (W/(m K))
θ angle coordinate
Θ dimensionless temperature

Subscripts
i inner boundary
o outer boundary

Several papers have been published on the determination of shape factors for more complicated cross-sections possessing regular polygons [4–8]. Empirical correlation for the arrangements of two concentric squares or a circle within the centre of a square is given by Smith et al. [4] in the form

$$S = \frac{2.79}{\ln\left(\frac{1}{E_{cp}}\right) + 0.036}. \tag{5}$$

They used electrical and thermal analogy and made experiments on the electrical flow through paper with constant conductivity. Because of the used method this formula possesses limited accuracy and has rather historical meaning.

Other workers developed approximate analytical solutions of the shape factor for hollow prismatic cylinders bounded by inner circles and outer regular polygons. Balcerzak and Raynor [5] based their solution obtained by point matching on the outer boundary, and have proposed the following formula:

$$S = \frac{2\pi}{\ln\left(\frac{1}{E_{cp}\cos\frac{\pi}{L}}\right) - V}, \tag{6}$$

where constant V is the function of number of polygonal sides L, and is given in Table 1.

Laura and Susemihl [6] have considered the same case of cross-section using conformal mapping in the solution of appropriate boundary value problem. They obtained the following formula:

$$S = \frac{2\pi}{\ln\left(\frac{V}{E_{cp}}\right)}, \tag{7}$$

where constant V is of the function of a number of polygonal sides L, and is given in Table 2. Formulas (6) and (7) give nearly identical results which are accurate only up to E_{cp}, i.e, about 0.8.

Table 1
Values of constant V in formula (6)

L	V
3	0.569580
4	0.270795
5	0.160686
6	0.106695
7	0.077607
8	0.056985
9	0.044160
10	0.035380

Table 2
Values of constant V in formula (7)

L	V
3	1.13209
4	1.07870
5	1.05246
6	1.03754

For the same geometry (for hollow prismatic cylinders bounded by inner circles and outer regular polygons) Simeza and Yovanovich [7] using the 'parallel flux tube' heat flow model have obtained the following formula for the shape factor:

$$S = 2L\left\{\frac{1}{V\sqrt{V^2+0.5^2}}tg^{-1}\left[\frac{\sqrt{V^2+0.5^2}}{V}tg\left(\frac{\pi}{L}\right)\right]\right\}, \quad (8)$$

where

$$V^2 = \ln\left(\frac{1}{E_{\rm cp}}\right). \quad (9)$$

The last equation is valid for cylinders having both small and large inner holes. The accuracy of the results improves with the increase of a hole size. Therefore, the expression gives accurate results for cylinders with large holes; this is the region, where the simple formulas, that existed before, namely (6) and (7), are inaccurate or fail.

Nickolay et al. [8] applied the method of finite elements and the method of finite differences to calculate the shape factors for cross-sections bounded by concentric circles and squares. These numerical methods are used to calculate the shape factors for three differently shaped cross-sections. These are:

- concentric squares,
- circle within the centre of a square,
- square within the centre of a circle.

By developing the shape factor for the investigated cross-sections analytical approximations for the shape factor are obtained in the forms

for concentric squares

$$S = \frac{8}{\ln\left(\frac{1}{E_{\rm pp}}\right)\left[1+\frac{1}{4}(1-E_{\rm pp})\right]}; \quad (10)$$

for a circle within the centre of a square

$$S = \frac{2\pi}{\ln u + 0.07577}\frac{1.0814u-1}{\sqrt{(u-1)(1.0814^2u-1)}}, \quad (11)$$

where $u = 1/E_{\rm cp}$;

for a square within the centre of a circle

$$S = \frac{2\pi}{\ln\left(\frac{p\pi}{4}\right)}\ln\left(\frac{p-1}{p-\sqrt{2}}\right)\frac{p-1.318}{0.4213}, \quad (12)$$

where $p = \sqrt{2}/E_{\rm pc}$.

The approximations (10)–(12) fulfill limited cases and are close to the exact solutions when being fitted to the numerical values, but unfortunately only for the case when one boundary is square.

From the above presented review it results that in some cases of cylinders whose cross-section has an inner or an outer contour in the form of regular polygon or a circle, the shape factor is known, but not for a general case. As a consequence, the purpose of this paper is to propose the simple analytical formulas for the shape factors for three cases of cylinders whose cross-section has an inner or an outer contour in the form of a regular polygon or circle. These three cases are the following: (a) hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons, (b) hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles, (c) hollow prismatic cylinders bounded by isothermal inner and outer regular polygons. In the existing literature only the case (a) is recognised relatively well. The boundary collocation method in the least squares sense for solving appropriate boundary value problems is used. By means of nonlinear approximation (Marquardt method), for the three considered geometry cases, the simple analytical formulas for the shape factors are obtained. A comparison of the obtained results to the above given results of other authors is presented.

2. Solution of the boundary value problem for temperature

In the considered cases the steady temperature field is governed by the two-dimensional Laplace equation. In recent years, numerical techniques relying on computer applications have seen the increasing use of finite differences, finite elements, and boundary elements. These methods, by their nature, offer approximate solutions only. However, these tools may prove too complex to use in the case of the simple boundary value problem. The time required to read the user's manual and to learn the procedure of mesh generating and supplying the input data can make these numerical method unreasonable to use (a sledgehammer solution to crack a simple nut).

One of the simplest numerical method which can be used for the solution of two-dimensional Laplace equation is a boundary collocation method which belongs to the family of boundary procedures. The boundary collocation method can be summarised as the method that consists in using the exact solutions to the governing differential equation of the problem and satisfying the given boundary conditions at a finite number of discrete points along the boundary. An extensive review of the boundary collocation method as used in linear continuous mechanics is given in [9].

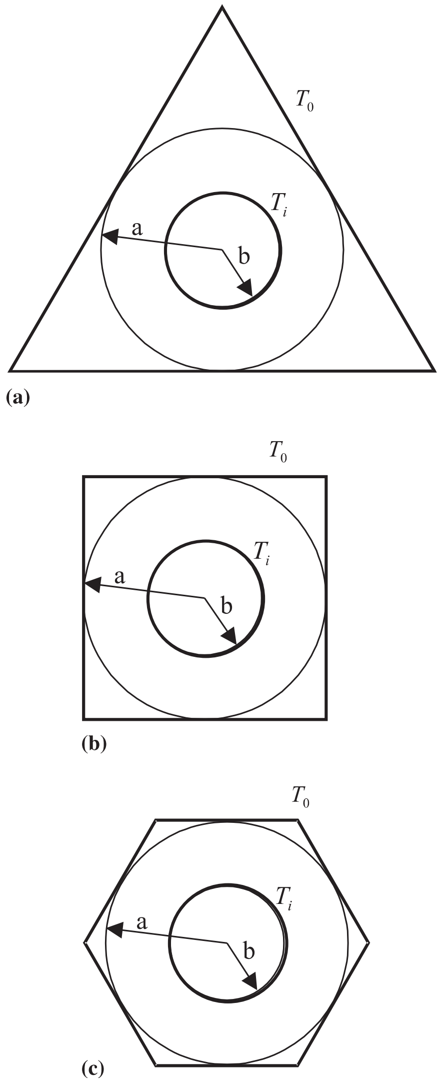

Fig. 1. Hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons.

In this paper, we show the application of the boundary collocation method for the treatment of domains with regular polygonal shapes. The shape of domains permits us to use special purpose trial functions [10].

For calculation reasons, in the problem below it is more convenient to use a polar coordinate system, in which the two-dimensional Laplace equation has the form

$$\frac{\partial^2 \Theta}{\partial R^2} + \frac{1}{R}\frac{\partial \Theta}{\partial R} + \frac{1}{R^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0. \tag{13}$$

When the collocation method is used for solving Eq. (13) the general solution is adopted in the form

$$\begin{aligned}\Theta(R,\theta) &= A + B\ln R + C\theta + D\theta\ln R \\ &\quad + \sum_{k=1}^{\infty}\left(A_k R^{\lambda_k} + B_k R^{-\lambda_k}\right)\cos(\lambda_k\theta) \\ &\quad + \sum_{k=1}^{\infty}\left(C_k R^{\lambda_k} + D_k R^{-\lambda_k}\right)\sin(\lambda_k\theta),\end{aligned} \tag{14}$$

where $A, B, C, D, A_k, B_k, C_k, D_k, \lambda_k$ are unknown constants.

In what follows, the application of the boundary collocation method is demonstrated by three examples, concerning: (a) temperature field for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons, (b) temperature field for hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles, (c) temperature field for hollow prismatic cylinders bounded by isothermal inner and outer regular polygons.

2.1. Temperature field for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons

Consider a family of long hollow regular prismatic cylinders of uniform thermal conductivity as shown in Fig. 1. The outer boundary is a regular polygon with L sides, while the inner boundary is a circle of radius b. The radius of the inscribed circle on a regular polygon is a.

The inner and outer boundaries of cylinders are maintained at uniform temperatures T_i and T_o.

Since the problem is symmetric with respect to the lines $\theta = 0$ and $\theta = \pi/L$, the solution of Eq. (1) can be found only between these two lines (see Fig. 2), where L is the number of polygon sides. The boundary conditions for Eq. (13) are the following:

$$T = T_\mathrm{i} \quad \text{for} \quad r = b, \tag{15}$$

$$T = T_\mathrm{o} \quad \text{for} \quad x = a, \tag{16}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \tag{17}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{18}$$

Let us introduce the nondimensional variables in the form

$$R = \frac{r}{a}, \quad E_\mathrm{cp} = \frac{b}{a}, \quad X = \frac{x}{a}. \tag{19}$$

Now, using (3), the formulation of the boundary value problem is the following:

(a) the governing Eq. (13),

(b) the boundary conditions.

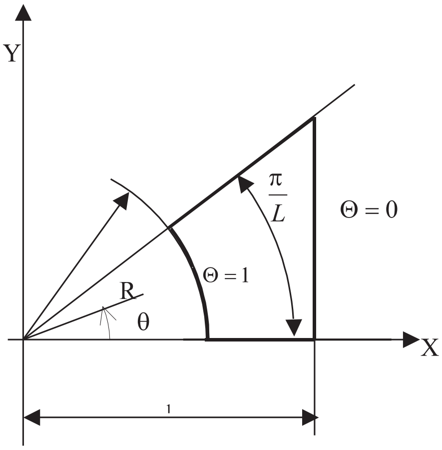

Fig. 2. Repeated element in hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons.

$$\Theta = 1 \quad \text{for} \quad R = E_{\mathrm{cp}}, \tag{20}$$

$$\Theta = 0 \quad \text{for}\ X = 1 \quad \text{or} \quad R_{\mathrm{b}} = \frac{1}{\cos\theta}, \tag{21}$$

$$\frac{\partial\Theta}{\partial\theta} = 0 \quad \text{for} \quad \theta = 0, \tag{22}$$

$$\frac{\partial\Theta}{\partial\theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{23}$$

Some of the constants in general solution (14) can be determined exactly.

From (22), we have

$$\frac{\partial\Theta}{\partial\theta}(\theta=0) = C + D\ln R + \sum_{k=1}^{\infty}\lambda_k\left[-\left(A_kR^{\lambda_k} + B_kR^{-\lambda_k}\right)\sin(\lambda_k 0) + \left(C_kR^{\lambda_k} + D_kR^{-\lambda_k}\right)\cos(\lambda_k 0)\right] = 0. \tag{24}$$

The last equation will be fulfilled if

$$C = D = C_k = D_k = 0, \quad k = 1, 2, \ldots \tag{25}$$

From (23) we have

$$\frac{\partial\Theta}{\partial\theta}\left(\theta = \frac{\pi}{L}\right) = \sum_{k=1}^{\infty}\lambda_k\left[\left(A_kR^{\lambda_k} + B_kR^{-\lambda_k}\right)\sin\left(\lambda_k\frac{\pi}{L}\right)\right] = 0. \tag{26}$$

Eq. (26) is satisfied when

$$\sin\left(\lambda_k\frac{\pi}{L}\right) = 0 \tag{27}$$

then

$$\lambda_k = kL. \tag{28}$$

Substituting the obtained results into (2), yields

$$\Theta = A + B\ln R + \sum_{k=1}^{\infty}\left(A_kR^{kL} + B_kR^{-kL}\right)\cos(kL\theta) \tag{29}$$

Using (20) in (29) gives

$$1 = A + B\ln E_{\mathrm{cp}} + \sum_{k=1}^{\infty}\left(A_kE_{\mathrm{cp}}^{kL} + B_kE_{\mathrm{cp}}^{-kL}\right)\cos(kL\theta). \tag{30}$$

Eq. (30) is satisfied if

$$A + B\ln E_{\mathrm{cp}} = 1, \tag{31}$$

$$A_kE_{\mathrm{cp}}^{kL} + B_kE_{\mathrm{cp}}^{-kL} = 0, \quad k = 1, 2, \ldots \tag{32}$$

or

$$A = 1 - B\ln E_{\mathrm{cp}}, \tag{33}$$

$$B_k = -A_kE_{\mathrm{cp}}^{2kL}, \quad k = 1, 2, \ldots \tag{34}$$

After using boundary conditions (20), (22) and (23), the solution adopts the form

$$\Theta = 1 + B\ln\frac{R}{E_{\mathrm{cp}}} + \sum_{k=1}^{\infty}A_k\left(R^{kL} - \frac{E_{\mathrm{cp}}^{2kL}}{R^{kL}}\right)\cos(kL\theta). \tag{35}$$

Introducing new symbols

$$Y_1 = B, \quad Y_2 = A_1, \quad Y_3 = A_2, \tag{36}$$

Eq. (35) can be written as

$$\Theta = 1 + Y_1\ln\frac{R}{E_{\mathrm{cp}}} + \sum_{k=2}^{\infty}Y_k\left(R^{(k-1)L} - \frac{E_{\mathrm{cp}}^{2(k-1)L}}{R^{(k-1)L}}\right)\cos[L(k-1)\theta]. \tag{37}$$

Solution (37) fulfils exactly governing Eq. (1) and boundary conditions (20), (22) and (23). Taking into account condition (21) and using (29) we have

$$Y_1\ln\frac{R_{\mathrm{b}}}{E_{\mathrm{cp}}} + \sum_{k=2}^{\infty}Y_k\left(R_{\mathrm{b}}^{(k-1)L} - \frac{E_{\mathrm{cp}}^{2(k-1)L}}{R_{\mathrm{b}}^{(k-1)L}}\right)\cos[L(k-1)\theta] = -1. \tag{38}$$

Choosing M values of angle θ given by the formula

$$\theta_j = tg^{-1}\left(\frac{j-1}{M-1}tg\frac{\pi}{L}\right), \quad j = 1, 2, 3, \ldots, M \tag{39}$$

and truncating infinite series in (38) to N first terms we obtain a system of linear equations for unknown constants Y_k

$$\sum_{k=1}^{N}G_{jk}Y_k = F_j, \quad j = 1, 2, \ldots, M, \tag{40}$$

where

$$G_{j1} = \ln\frac{R_{bj}}{E_{\mathrm{cp}}}, \tag{41}$$

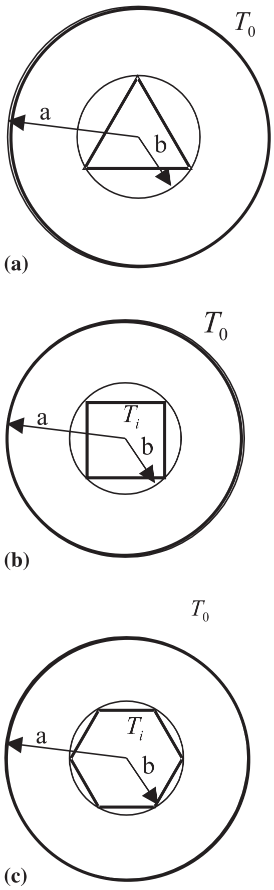

Fig. 3. Hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles.

$$G_{jk} = \left(R_{bj}^{(k-1)L} - \frac{E_{cp}^{2(k-1)L}}{R_{bj}^{(k-1)L}} \right) \cos\left[L(k-1)\theta_j \right], \tag{42}$$

$$F_j = -1, \tag{43}$$

$$R_{bj} = \frac{1}{\cos\theta_j}. \tag{44}$$

The system of linear Eq. (40) results from the simplest version of the boundary collocation method in which the number of collocation points M must be equal to the number of unknown coefficients N. This version is named hereafter: simple boundary collocation method (SBCM). As it will be shown in paragraph 4, such a version does not guarantee good results for all values of parameters L and E_{cp}. We have better results when we use the boundary collocation method in the least squares sense (BCMLSS). It leads to the following system of equations for unknown coefficients:

$$\sum_{k=1}^{N} H_{jk} Y_k = P_j, \quad j = 1, 2, \ldots, N, \tag{45}$$

where

$$H_{ij} = \sum_{k=1}^{M} G_{ki} G_{kj}, \tag{46}$$

$$P_i = \sum_{k=1}^{M} G_{ki} F_k. \tag{47}$$

Now, the number of collocation points M can be equal to or greater than the number of unknown coefficients N ($M \geqslant N$).

2.2. Temperature field for hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles

Consider a family of long hollow regular prismatic cylinders of uniform thermal conductivity as shown in Fig. 3. The outer boundary is a circle of radius b, while the inner boundary is a regular polygon with L sides. The radius of the circumscribed circle on a regular polygon is a.

Again, since the problem is symmetrical with respect to the lines $\theta = 0$ and $\theta = \pi/L$, the solution of Eq. (13) can be found only between these two lines (see Fig. 4), where L is the number of polygon sides. The boundary conditions for Eq. (13) are the following:

$$T = T_i \quad \text{for} \quad x = b\cos\frac{\pi}{L}, \tag{48}$$

$$T = T_o \quad \text{for} \quad r = a, \tag{49}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \tag{50}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{51}$$

Let us introduce the nondimensional variables in the form

$$R = \frac{r}{a}, \quad E_{cp} = \frac{b}{a}, \quad X = \frac{x}{a}. \tag{52}$$

Now, using (3), the formulation of the boundary value problem is the following: the governing Eq. (13) and the boundary conditions

$$\Theta = 0 \quad \text{for} \quad R = 1, \tag{53}$$

$$\Theta = 1 \quad \text{for} \quad X = E_{pc}\cos\frac{\pi}{L} \quad \text{or} \quad R_b = \frac{E_{pc}\cos\frac{\pi}{L}}{\cos\theta}, \tag{54}$$

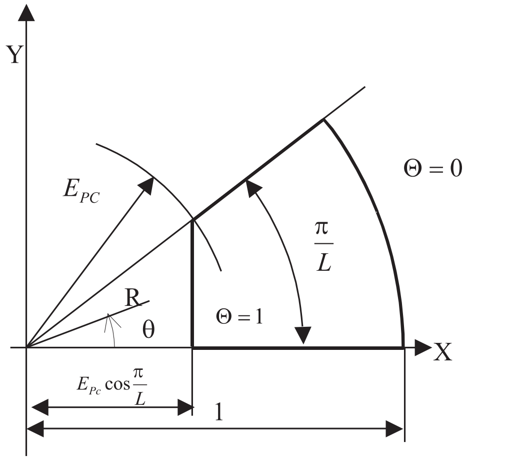

Fig. 4. Repeated element in hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles.

$$\frac{\partial \Theta}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \tag{55}$$

$$\frac{\partial \Theta}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{56}$$

Taking into account general solution (14) and fulfilling exactly boundary conditions (48), (50) and (51) in the same way as in the previous case one has

$$\Theta = B \ln R + \sum_{k=1}^{\infty} A_k \left(R^{kL} - \frac{1}{R^{kL}} \right) \cos (kL\theta). \tag{57}$$

Introducing new symbols

$$Y_1 = B, \quad Y_2 = A_1, \quad Y_3 = A_2, \tag{58}$$

Eq. (52) can be written as

$$\Theta = Y_1 \ln R + \sum_{k=2}^{\infty} Y_k \left(R^{(k-1)L} - \frac{1}{R^{(k-1)L}} \right) \cos [L(k-1)\theta]. \tag{59}$$

Solution (54) fulfils exactly governing Eq. (2) and boundary conditions (48), (50) and (51). Taking into account condition (49) we have

$$Y_1 \ln R_\mathrm{b} + \sum_{k=2}^{\infty} Y_k \left(R_\mathrm{b}^{(k-1)L} - \frac{1}{R_\mathrm{b}^{(k-1)L}} \right) \cos [L(k-1)\theta] = 1. \tag{60}$$

Choosing M values of angle θ given by formula (39) and truncating infinite series in (54) to N first terms we obtain a system of linear equations in the form (40) for unknown constants Y_k in which

$$G_{j1} = \ln R_{bj}, \tag{61}$$

$$G_{jk} = \left(R_{bj}^{(k-1)L} - \frac{1}{R_{bj}^{(k-1)L}} \right) \cos \left[L(k-1)\theta_j \right], \tag{62}$$

$$F_j = 1. \tag{63}$$

In the case, when $N = M$ one has the SBCM. For boundary collocation method in the least squares sense one can use (45)–(47) with matrix and vector given by (56)–(58).

2.3. Temperature field for hollow prismatic cylinders bounded by isothermal inner and outer regular polygons

Consider a family of long hollow prismatic cylinders bounded by isothermal inner and outer regular polygones as shown in Fig. 5. The radius of the circumscribed circle on the inner regular polygon is b, whereas on the outer polygon is a. The inner and outer boundaries of cylinders are maintained at uniform temperatures T_i and T_o.

Again, since the problem is symmetrical with respect to the lines $\theta = 0$ and $\theta = \pi/L$, the solution of Eq. (46)

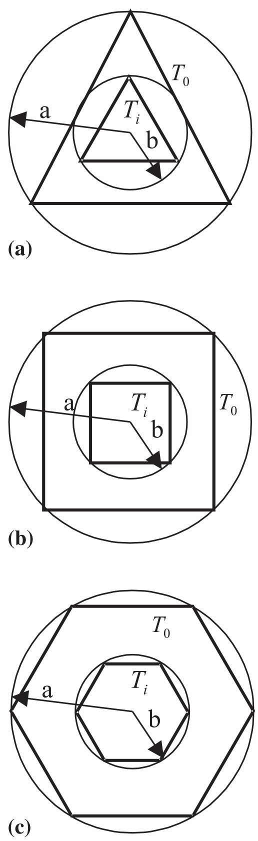

Fig. 5. Hollow prismatic cylinders bounded by isothermal inner and outer regular polygons.

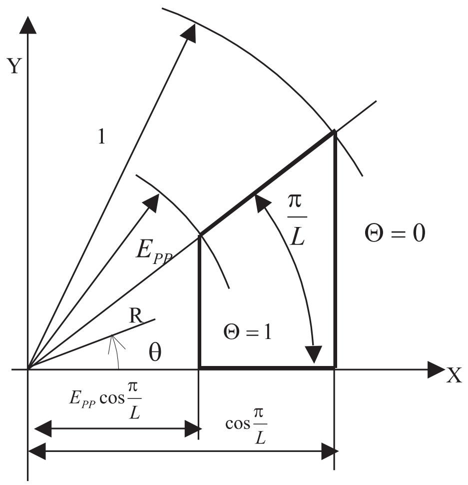

Fig. 6. Repeated element in hollow prismatic cylinders bounded by isothermal inner and outer regular polygons.

can be found only between these two lines (see Fig. 6), where L is the number of polygon sides. The boundary conditions for Eq. (14) are the following:

$$T = T_i \quad \text{for} \quad x = b\cos\frac{\pi}{L}, \tag{64}$$

$$T = T_o \quad \text{for} \quad x = a\cos\frac{\pi}{L}, \tag{65}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \tag{66}$$

$$\frac{\partial T}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{67}$$

Let us introduce the nondimensional variables in the form

$$R = \frac{r}{a}, \quad E_{pp} = \frac{b}{a}, \quad X = \frac{x}{a}. \tag{68}$$

Now the formulation of the boundary value problem is the following:

(a) the governing Eq. (13),
(b) the boundary conditions.

$$\Theta = 1 \quad \text{for} \quad X = E_{pp}\cos\frac{\pi}{L}, \tag{69}$$

$$\Theta = 0 \quad \text{for} \quad X = \cos\frac{\pi}{L}, \tag{70}$$

$$\frac{\partial \Theta}{\partial \theta} = 0 \quad \text{for} \quad \theta = 0, \tag{71}$$

$$\frac{\partial \Theta}{\partial \theta} = 0 \quad \text{for} \quad \theta = \frac{\pi}{L}. \tag{72}$$

Taking into account general solution (14) and fulfilling exactly boundary conditions (71) and (72) in the same way as in the previous case one has

$$\Theta = A + B\ln R + \sum_{k=1}^{\infty}\left(A_k R^{kL} + B_k R^{-kL}\right)\cos(kL\theta). \tag{73}$$

Truncating infinite series in (73) to the first $2N$ terms and introducing notations

$$A = Y_1,\ A_2 = Y_2,\ A_3 = Y_3, \ldots,\ A_N = Y_N,$$
$$B = Y_{N+1},\ B_2 = Y_{N+2},\ B_3 = Y_{N+3}, \ldots,\ B_N = Y_{2N} \tag{74}$$

one has

$$\Theta = \sum_{k=1}^{2N} Y_k \varphi_k(R, \theta, L), \tag{75}$$

where

$$\varphi_1 = 1,$$
$$\varphi_k = R^{(k-1)L}\cos[(k-1)L\theta] \quad \text{for} \quad k = 2, 3, \ldots, N,$$
$$\varphi_{N+1} = \ln R, \tag{76}$$
$$\varphi_k = R^{-(k-N-1)L}\cos[(k-N-1)L\theta]$$
$$\text{for} \quad k = N+2, N+3, \ldots, 2N.$$

Introducing the collocation points given by

$$R_{b1j} = \frac{E_{pp}\cos\frac{\pi}{L}}{\cos\theta_j},$$

$$R_{b2j} = \frac{\cos\frac{\pi}{L}}{\cos\theta_j},$$

where θ_j is given by (39), one has the system of linear equations in the following form:

$$\sum_{k=1}^{2N} G_{jk} Y_k = F_j, \quad j = 1, 2, \ldots, 2M, \tag{77}$$

where

$$G_{jk} = \varphi_k(R_{b1k}, \theta_k, L), \tag{78}$$

$$G_{N+j,N+k} = \varphi_{N+k}\left(R_{b2j}, \theta_j, L\right), \tag{79}$$

$$F_k = 1.0, \tag{80}$$

$$F_{k+N} = 0. \tag{81}$$

Now again, in the case when $N = M$ one has the SBCM. For boundary collocation method in the least squares sense one can use (45)–(47) with matrix and vector given by (78)–(81).

Table 3
Values of shape factor S for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons according to Fig. 1[a]

E_{cp}	L	S	ERL	ELG
0.10	3	2.5892417837	0.10224×10^{-3}	0.26085×10^{-8}
0.10	4	2.6418293009	0.20945×10^{-6}	0.12045×10^{-13}
0.10	5	2.6694659434	0.14482×10^{-3}	0.14653×10^{-8}
0.10	6	2.6857476185	0.25221×10^{-3}	0.37600×10^{-8}
0.10	8	2.7031680163	0.31068×10^{-3}	0.44294×10^{-8}
0.10	10	2.7118023127	0.28538×10^{-3}	0.30832×10^{-8}
0.10	16	2.7217688063	0.20507×10^{-3}	0.76080×10^{-9}
0.10	32	2.7269250701	0.77953×10^{-4}	0.46865×10^{-10}
0.10	Circle	2.7287528013	–	–
0.20	3	3.6245639717	0.14330×10^{-3}	0.51252×10^{-8}
0.20	4	3.7284525350	0.29577×10^{-6}	0.24020×10^{-13}
0.20	5	3.7837372069	0.20527×10^{-3}	0.29438×10^{-8}
0.20	6	3.8165315639	0.35840×10^{-3}	0.75928×10^{-8}
0.20	8	3.8518054114	0.44270×10^{-3}	0.89936×10^{-8}
0.20	10	3.8693603554	0.40720×10^{-3}	0.62771×10^{-8}
0.20	16	3.8896833143	0.29306×10^{-3}	0.15538×10^{-8}
0.20	32	3.9002226886	0.11149×10^{-3}	0.95870×10^{-10}
0.20	Circle	3.9039626764	–	–
0.30	3	4.7312803635	0.18944×10^{-3}	0.89787×10^{-8}
0.30	4	4.9097637284	0.39486×10^{-6}	0.42890×10^{-13}
0.30	5	5.0060779483	0.27158×10^{-3}	0.51531×10^{-8}
0.30	6	5.0636441648	0.47551×10^{-3}	0.13366×10^{-7}
0.30	8	5.1259250419	0.58915×10^{-3}	0.15928×10^{-7}
0.30	10	5.1570615962	0.54272×10^{-3}	0.11150×10^{-7}
0.30	16	5.1932253440	0.39127×10^{-3}	0.27698×10^{-8}
0.30	32	5.2120295832	0.14899×10^{-3}	0.17121×10^{-9}
0.30	Circle	5.2187106443	–	–
0.40	3	6.0403996037	0.25736×10^{-3}	0.16759×10^{-7}
0.40	4	6.3336147525	0.57434×10^{-6}	0.92124×10^{-13}
0.40	5	6.4947316743	0.35232×10^{-3}	0.86727×10^{-8}
0.40	6	6.5919495086	0.61903×10^{-3}	0.22651×10^{-7}
0.40	8	6.6978912107	0.76983×10^{-3}	0.27195×10^{-7}
0.40	10	6.7511525323	0.71049×10^{-3}	0.19109×10^{-7}
0.40	16	6.8132633966	0.51332×10^{-3}	0.47675×10^{-8}
0.40	32	6.8456662538	0.19569×10^{-3}	0.29535×10^{-9}
0.40	Circle	6.8571964832	–	–
0.50	3	7.6944300913	0.39639×10^{-3}	0.41029×10^{-7}
0.50	4	8.1724712686	0.12058×10^{-5}	0.42505×10^{-12}
0.50	5	8.4420455222	0.45779×10^{-3}	0.14642×10^{-7}
0.50	6	8.6069274206	0.80819×10^{-3}	0.38609×10^{-7}
0.50	8	8.7884026183	0.10101×10^{-2}	0.46820×10^{-7}
0.50	10	8.8803271117	0.93458×10^{-3}	0.33064×10^{-7}
0.50	16	8.9881056247	0.67717×10^{-3}	0.82970×10^{-8}
0.50	32	9.0445824146	0.25855×10^{-3}	0.51557×10^{-9}
0.50	Circle	9.0647205359	–	–
0.60	3	9.9260297930	0.75982×10^{-3}	0.15883×10^{-6}
0.60	4	10.7181846200	0.40527×10^{-5}	0.50312×10^{-11}
0.60	5	11.1823645782	0.60513×10^{-3}	0.25581×10^{-7}
0.60	6	11.4724411603	0.10765×10^{-2}	0.68499×10^{-7}
0.60	8	11.7967989139	0.13558×10^{-2}	0.84347×10^{-7}
0.60	10	11.9629998640	0.12590×10^{-2}	0.60003×10^{-7}
0.60	16	12.1594193027	0.91608×10^{-3}	0.15185×10^{-7}
0.60	32	12.2630109133	0.35055×10^{-3}	0.94777×10^{-9}
0.60	Circle	12.3000602915	–	–

Table 3 (*continued*)

E_{cp}	L	S	ERL	ELG
0.70	3	13.2052694481	0.18546×10^{-2}	0.10039×10^{-5}
0.70	4	14.5734159748	0.17139×10^{-4}	0.92693×10^{-10}
0.70	5	15.4214667694	0.82676×10^{-3}	0.47727×10^{-7}
0.70	6	15.9705681529	0.14921×10^{-2}	0.13160×10^{-6}
0.70	8	16.6022454901	0.19064×10^{-2}	0.16675×10^{-6}
0.70	10	16.9328087961	0.17818×10^{-2}	0.12017×10^{-6}
0.70	16	17.3289384805	0.13055×10^{-2}	0.30842×10^{-7}
0.70	32	17.5401025858	0.50139×10^{-3}	0.19390×10^{-8}
0.70	Circle	17.6159982326	–	–
0.80	3	18.7353581558	0.58079×10^{-2}	0.10543×10^{-4}
0.80	4	21.3062946852	0.89455×10^{-4}	0.26608×10^{-8}
0.80	5	23.0356625040	0.11909×10^{-2}	0.98794×10^{-7}
0.80	6	24.2232845716	0.22150×10^{-2}	0.28999×10^{-6}
0.80	8	25.6653644633	0.29238×10^{-2}	0.39216×10^{-6}
0.80	10	26.4546863315	0.27758×10^{-2}	0.29158×10^{-6}
0.80	16	27.4313266797	0.20662×10^{-2}	0.77265×10^{-7}
0.80	32	27.9641865514	0.79935×10^{-3}	0.49287×10^{-8}
0.80	Circle	28.1575957029	–	–
0.90	3	31.2585633122	0.27783×10^{-1}	0.27529×10^{-3}
0.90	4	37.1852486539	0.98922×10^{-3}	0.38887×10^{-6}
0.90	5	41.6921039904	0.18335×10^{-2}	0.22334×10^{-6}
0.90	6	45.1220719865	0.37069×10^{-2}	0.81032×10^{-6}
0.90	8	49.7760507303	0.53395×10^{-2}	0.13071×10^{-5}
0.90	10	52.6136499028	0.53350×10^{-2}	0.10762×10^{-5}
0.90	16	56.4813122579	0.42239×10^{-2}	0.32308×10^{-6}
0.90	32	58.7742191736	0.16797×10^{-2}	0.21768×10^{-7}
0.90	Circle	59.6350906505	–	–

[a] Number of series terms in Eq. (45) $N = 6$. Number of collocation points $M = 20$.

3. Shape factors

For all the considered cases of geometry the shape factor defined by (2), after some calculations, can be expressed as:

$$S = \int_{\Gamma} \frac{\partial \Theta}{\partial \mathbf{n}} \, d\Gamma = -\int_0^{2\pi} \frac{\partial \Theta}{\partial R} R \, d\theta = -2\pi Y_1. \tag{82}$$

The last result means that only coefficient Y_1 at the logarithmic term is responsible for the shape factor. On the other hand the coefficients Y_k obtained from Eqs. (40), (45), or (77), respectively are functions of four parameters: L, E, N, M, where E is defined in an appropriate way for each case. The first two parameters have physical meaning, while the other two (N, M) result from the chosen numerical method. In the next section, it will be shown that parameters N and M, after reasonable choice, have practically marginal influence on the shape factor.

An example of calculated values of shape the factor according to formula (82) is presented in Table 3. These values can be used to calculate the heat flow according to Eq. (1) but it is a little bit inconvenient. A better possibility is an approximate formula, which covers all values from Table 3. Here we propose the approximate formula in the following form:

$$S^* = f(L, E, \mathbf{k}) = \frac{2\pi}{\ln(1/E) + \ln\{k_1 \exp[k_2(L - k_3)] + 1\}}, \tag{83}$$

where k_1, k_2, and k_3 are appropriately chosen constants.

To determine coefficients k_1, k_2, and k_3 the method of the least squares minimisation is adapted. Taking into account values obtained by means of the boundary collocation method (an example is given in Table 3), and using the Marquardt method [11] for the solution of nonlinear equations resulting from the least squares minimisation one can obtain the results presented in Table 4.

4. Results and conclusions

Two different error criteria in the estimate of exactness of the boundary collocation method in the solution of boundary value problems have been employed. The first one is based on 'global' error measure for Θ and given by

Table 4
Values of coefficients k_1, k_2, and k_3 in formula (83)

	Hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons	Hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles	Hollow prismatic cylinders bounded by isothermal inner and outer regular polygons
k_1	0.1267244657714413	0.5381454843892344	−0.0669303436356801
k_2	−0.3476423728952475	−0.2902395077433192	−0.3010864083155045
k_3	2.6537510672908366	0.0315201822595446	−0.9220961055481248

$$ERG = \frac{1}{NP}\sum_{i=1}^{NP}[\Theta_e(R_i, \theta_i) - \Theta_a(R_i, \theta_i)]^2. \quad (84)$$

The subscripts e and a above refer to the exact value resulting from the boundary conditions and approximate solution obtained by the boundary collocation method. Points (R_i, θ_i) at which an error is evaluated are uniformly distributed over a part of the boundary, where the boundary condition is fulfilled approximately. NP is the number of nodes at that boundary (in calculations $NP = 1000$ was taken).

The second error criterion has a local character and is defined by

$$ERL = \max|\Theta_e(R_b, \theta_b) - \Theta_a(R_b, \theta_b)|, \quad (85)$$

where R_b, θ_b are coordinates of the boundary, where the boundary condition is fulfilled approximately. In the boundary collocation method the local criterion is a maximum error of the method because for the elliptic problem the maximum error occurs on the boundary [12].

The proposed determination of the shape factor is based on the solution of the appropriate boundary value problem. The analytical–numerical method has been used for the solution of this boundary value problem. In this method the governing equation and boundary condition at some part of the boundary are satisfied exactly. Only the boundary condition at the remaining part of the boundary is satisfied approximately by means of boundary collocation method. In preliminary calculations two different methods of boundary collocation have been used. In the first one – the SBCM, the boundary condition is satisfied in the finite number of collocation points exactly. In such version the number of terms in the series in the assumed form of the solution is equal to the number of collocation points at the above mentioned remaining part of boundary. In the second case – the BCMLSS, the boundary condition is fulfilled in the least squares sense in some number of points greater than the number of terms in series. In preliminary calculations both methods have been used. When we use the SBCM, it can be seen intuitively that increasing the number of collocation points leads to more exact results. It is true, but not for all cases. Among others, Table 5 illustrates the case when maximal error

Table 5
Values of maximal error ERL and shape factor S – numbers in brackets, versus number of collocation points M and number of terms in series N for $L = 3$[a]

N	$E_{cp} = 0.8$		$E_{cp} = 0.85$		$E_{cp} = 0.9$	
	SBCM $M = N$	BCMLSS $M = 20$	SBCM $M = N$	BCMLSS $M = 20$	SBCM $M = N$	BCMLSS $M = 20$
3	0.1388 (19.14)	0.05525 (18.69)	0.2200 (24.32)	0.08959 (23.22)	0.4034 (33.75)	0.1684 (30.34)
4	0.07483 (18.77)	0.02279 (18.73)	0.1325 (23.53)	0.03823 (23.38)	0.2738 (31.72)	0.07824 (31.04)
5	0.05141 (18.74)		0.1003 (23.43)		0.2327 (31.37)	
6	0.03927 (18.73)	0.005808 (18.73)	0.083393 (23.41)	0.01175 (23.41)	0.2197 (31.27)	0.02778 (31.26)
7	0.03167 (18.73)		0.07433 (23.41)		0.2245 (31.26)	
8	0.02646 (18.73)	0.003673 (18.73)	0.06861 (23.41)	0.008848 (23.41)	0.2343 (31.26)	0.02677 (31.26)
10	0.01998 (18.73)	0.006141 (18.73)	0.6425 (23.41)	0.008848 (23.41)	0.2916 (31.26)	0.07360 (31.26)
12	0.01637 (18.73)	0.02119 (18.73)	0.06608 (23.41)	0.08133 (23.41)	0.3999 (31.26)	0.4557 (31.26)
15	0.01359 (18.73)		0.07750 (23.41)		0.7181 (31.26)	
20	0.01217 (18.73)	11.98 (18.80)	0.1215 (23.41)	59.94 (23.79)	2.226 (31.26)	442.4 (34.45)
25	0.01230 (18.73)		0.2141 (23.41)		8.128 (31.25)	

[a] Hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons.

Table 6
Comparison of the results of the shape factor for a circle within the centre of a square ($L = 4$)

E_{cp}	Smith et al. [4]	Balcerzak and Raynor [5]	Laura and Susemilh [6]	Simeza et al. [7]	Nickolay et al. [8]	Our formula (83)
0.10	2.69305	2.64181	2.64183	2.62786	2.64273	2.64183
0.15	3.24453	3.18475	3.18478	3.16527	3.18659	3.18478
0.20	3.79607	3.72841	3.72846	3.70280	3.73162	3.72845
0.25	4.37263	4.29745	4.29751	4.26492	4.30263	4.29750
0.30	4.99214	4.90969	4.90978	4.86930	4.91774	4.90976
0.35	5.67152	5.58207	5.58218	5.53277	5.59425	5.58218
0.40	6.42946	6.33341	6.33355	6.27408	6.35162	6.33361
0.45	7.28864	7.18665	7.18683	7.11620	7.21374	7.18709
0.50	8.27819	8.17138	8.17162	8.08908	8.21181	8.17247
0.55	9.43724	9.32755	9.32786	9.23353	9.38845	9.33033
0.60	10.82030	10.71112	10.71152	10.60764	10.80435	10.71819
0.65	12.50635	12.40360	12.40414	12.29794	12.54987	12.42135
0.70	14.61483	14.52916	14.52990	14.44026	14.76677	14.57342
0.75	17.33579	17.28711	17.28816	17.26274	17.69243	17.39816
0.80	20.99165	21.01947	21.02102	21.18472	21.76104	21.30630
0.85	26.17726	26.36698	26.36942	27.08480	27.87461	27.15810
0.90	34.12531	34.68706	34.69129	37.23823	38.33590	37.18525

increases if the number of collocation points increases for the SBCM (see values of ERL for $E_{cp} = 0.9$). We cannot observe such phenomena when we use the BCMLSS. Numerical results from Table 5 indicate that the BCMLSS is very efficient, with 3 to 4 figures accuracy achievable for the shape factor by solving fewer than eight linear equations.

On the base presented of the results one can conclude that the BCMLSS is an ideal method for solving two-dimensional potential boundary value problems whose boundaries have regular polygonal shapes. It has the advantage of giving a solution in an explicit formula for potential (temperature) and for integral characteristic (shape factors), unlike the traditional numerical methods such as the FEM and the FDM when solution is given in number of points. The only numerical matter in the application of this method is the Gauss elimination procedure for calculation of the expansion coefficients related with a given problem.

As it was shown in Section 1, there are a few formulas for the shape factor given by other authors. The comparison of the results obtained on the base of these formulas to the results proposed in this paper are given in Tables 6–8. On the base of these comparisons one can give the following conclusions:

1. The proposed formula (83) for cross-section with regular polygonal shapes is the most general among the existing ones. Formulas given by other authors are good in some regions given in Section 1.

Table 7
Comparison of the results of the shape factor given in literature for hollow prismatic cylinders bounded by isothermal inner circles and outer regular polygons

L	E_{cp}	Sinth et al. [4]	Balcerzak and Raynor [5]	Laura and Sumemihl [6]	Simeza and Yovanovich [7]	Our formula (83)
3	0.3		4.73295	4.73118	4.68739	4.73128
4	0.3	4.99214	4.90969	4.90978	4.86930	4.90976
5	0.3		5.00563	5.00611	4.97627	5.00608
6	0.3		5.06251	5.06372	5.04228	5.06364
3	0.5		7.69325	7.68856	7.62854	7.69443
4	0.5	8.27819	8.17138	8.17162	8.08908	8.17247
5	0.5		8.44064	8.44199	8.37307	8.44205
6	0.5		8.60365	8.60711	8.55421	8.60693

Table 8
Comparison of the values of shape factor for a square within the centre of a circle and concentric squares

E_{pc} or E_{pp}	Square within the centre of a circle		Concentric squares	
	Nickolay et al. [8]	Our formula (83)	Nickolay et al. [8]	Our formula (83)
0.10	2.54406	2.52254	2.83621	2.83147
0.15	3.04449	3.01288	3.47787	3.46387
0.20	3.53653	3.49482	4.14223	4.11554
0.25	4.04204	3.98980	4.85960	4.81756
0.30	4.57541	4.51187	5.65504	5.59547
0.35	5.14910	5.07303	6.55513	6.47612
0.40	5.77585	5.68548	7.59205	7.49147
0.45	6.47023	6.36298	8.80764	8.68243
0.50	7.24995	7.12211	10.25916	10.10411
0.55	8.13771	7.98385	12.02838	11.83405
0.60	9.16370	8.97572	14.23720	13.98651
0.65	10.36966	10.13520	17.07663	16.73853
0.70	11.81577	11.51519	20.86454	20.38125
0.75	13.59334	13.19363	26.17268	25.43111
0.80	15.85093	15.29106	34.14416	32.90523

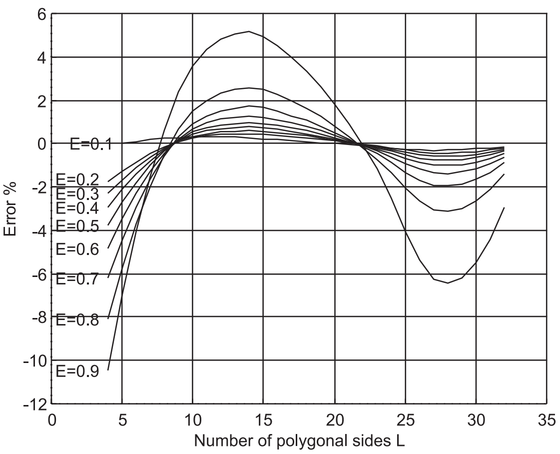

Fig. 7. Percentage error $\text{Error\%} = ((S_{col} - S_{appr})/S_{cal})\ 100$ of approximation given by formula (83), where S_{col} is the value of the shape factor obtained by means of boundary collocation method, S_{appr} is the value of the shape factor given by the proposed approximate formula (83). Hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles.

2. The results given by formula (83) are in good agreements with the results given by the existing formulas in a region where they are valid.

The purpose of this paper is to propose the simple analytical formula for the shape factor (formula (83)). This formula has been obtained by means of Marquardt method for the solution of the nonlinear approximation problem. The comparison of values of the shape factor obtained by means of the boundary collocation method S_{col} and given by the proposed approximate formula (83) S_{appr} for hollow prismatic cylinders bounded by isothermal inner regular polygons and outer circles is given in Fig. 7. One can observe that differences between 'exact' values of the shape factor and the approximate one are not essential from the engineering point of view.

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